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CONDITIONS FOR THE ONSET OF SLIDING IN A PLANE SYSTEM WITH FRICTION†

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The problem of the plane motion of a rigid body along a fixed surface in the presence of dry (Coulomb) friction is considered. The constraint is assumed to be non-restraining. It is shown that the validity of a certain system of two inequalities of the same type guarantees that the surfaces maintain contact and that the body will continue to roll without sliding. These conditions are analysed in a few specific cases of mechanical systems.

1. THE CONDITIONS THAT ENSURE PURE ROLLING OF A BODY

We shall consider the plane-parallel motion of a rigid body which can roll and slide along a fixed surface. Let us suppose that the contour of the body is represented in the plane of the motion, which passes through the centre of mass G of the body, by a regular closed convex curve PS, while PS_1 is the support curve; at each instant the curves are in contact at a single point.

Let us introduce a moving system of coordinates Pxy in which the axis Py is directed into the moving body; let (ξ, η) denote the coordinates of the centre G, m, ω and $\sigma^{1/2}$ the mass, instantaneous angular velocity and central radius of inertia of the body, respectively, and k_1 and k the curvatures (with signs) of the curves PS_1 and PS, respectively. Throughout, coordinates of vector quantities will be specified in the system Pxy.

If the instantaneous velocity of the point P of the body is not zero, then the total reaction $\mathbf{R}(F, N)$ exerted on the body by the support surface has a normal component $N > 0$ and a tangential component $F = fN$, where f $(0 < f < 1)$ is the coefficient of dry friction. In the case of pure rolling the instantaneous velocity of the point of contact of the body and the curve PS_1 is a constant, equal to zero, and the reaction $\mathbf{R}$ lies within the double angle of friction $M'PM$ (Fig. 1). Consequently, the following inequalities hold

$$\operatorname{mom}_A(\mathbf{R}) < 0, \quad \operatorname{mom}_{A'}(\mathbf{R}) > 0 \tag{1.1}$$

where A and A' are arbitrary points on the rays PM and PM', respectively. Later we shall specify the choice of A and A'.

When the body is rolling without sliding, the point P is the instantaneous centre of velocities, with acceleration $\mathbf{a}_p = (0, \omega^2(k - k_1)^{-1}$ [1, 2]. The principal vector of forces of inertia is $-m\mathbf{a}_G = m(\omega^2\xi, \omega^2(\eta - (k - k_1)^{-1})$, and the principal moment of forces of inertia is $-m\sigma\dot{\omega}$.

By d'Alembert's principle, the reaction $\mathbf{R}$, forces of inertia and active forces exerted on the body always form a zero system. In particular, this implies that the sum of the moments of these forces about an arbitrary point $A(x, y)$ in the plane must vanish

$$m(x\xi + y\eta - \xi^2 - \eta^2 - \sigma)\dot{\omega} + m\omega^2\left(y\xi - x\eta + \frac{x - \xi}{k - k_1}\right) + \Sigma_A + \operatorname{mom}_A(\mathbf{R}) = 0 \tag{1.2}$$

(Σ_A is the principal moment about A of the active forces exerted on the body). If A is taken on the line of critical poles [3]

$$x\xi + y\eta = \xi^2 + \eta^2 + \sigma \tag{1.3}$$

then the angular acceleration of the body does not occur in Eq. (1.2).

We will take as A and A' in inequalities (1.1) the points at which the straight line (1.3) cuts the rays

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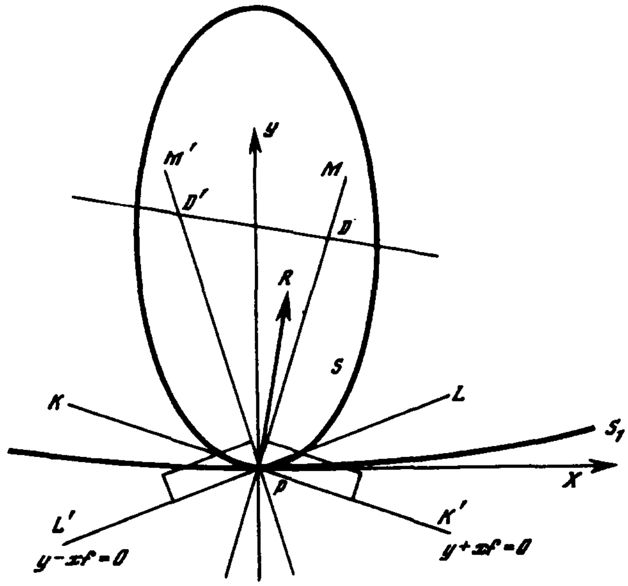

Fig. 1.

PM and *PM'*. Denote these points by *D* and *D'*. We have

$$y_D = \frac{\xi^2 + \eta^2 + \sigma}{\eta + \xi f}, \quad x_D = y_D f$$

The coordinates of *D'* are obtained by replacing *f* by $-f$ in these formulae.

Theorem. The body will continue to roll without sliding if and only if the points *D* and *D'*—provided both are in the finite part of the plane—satisfy the conditions

$$(\eta + \xi f)\,\mathrm{mom}_D(\mathbf{R}) < 0, \quad (\eta - \xi f)\,\mathrm{mom}_{D'}(\mathbf{R}) > 0 \tag{1.4}$$

Proof. The straight lines $y + xf = 0$ (*KK'*) and $y - xf = 0$ (*LL'*) divide the plane into four domains (Fig. 1). When the centre of mass $G(\xi, \eta)$ lies inside the angle *LPK*, inequalities (1.4) follow readily from conditions (1.1).

If the centre of mass falls on the ray *PK*, the line (1.3) of critical poles becomes parallel to the straight line $x - yf = 0$ and the point *D* goes to infinity. If the centre of mass falls inside the angle *KPL'*, then $\eta - \xi f > 0$, $\eta + \xi f < 0$. The constraint is strained, and pure rolling will continue if and only if $\mathrm{mom}_D(\mathbf{R}) > 0$, $\mathrm{mom}_{D'}(\mathbf{R}) > 0$, implying the truth of (1.4).

The remaining two cases, in which the centre of mass lies within the angle *L'PK'* or the angle *LPK'*, are dealt with in a similar fashion.

Note that when one of relations (1.4) becomes an equality, the line of action of the reaction **R** will coincide with the appropriate side *PM* or *PM'* of the double angle of friction. At the next instant the body will begin to slide although, theoretically speaking, it may still be possible that rolling will continue. To bring an issue to a close the analysis must be performed anew each time. However, in view of the fact that the limiting position of the vector **R** depends on the friction coefficient of *f*, whose numerical values are determined experimentally and not very accurately, such an analysis is of little interest in applications. We will therefore assume that, when the force **R** reaches its limiting direction, the body immediately begins to slide.

Finally, if both relations (1.4) become equalities, then $\mathbf{R} = 0$, i.e. the constraint is not imposed. This completes the proof of the theorem.

Applying (1.2) to the case in which the point *A* is put equal to each of the points *D* and *D'* in turn, we can rewrite the inequalities (1.4) as

$$m\omega^2(\eta+\xi f)\left(y_D\xi - x_D\eta + \frac{x_D-\xi}{k-k_1}\right)+(\eta+\xi f)\Sigma_D > 0 \tag{1.5}$$

$$m\omega^2(\eta-\xi f)\left(y_{D'}\xi - x_{D'}\eta + \frac{x_{D'}-\xi}{k-k_1}\right)+(\eta-\xi f)\Sigma_{D'} < 0$$

It is of some interest to analyse these conditions for specific examples of mechanical systems.

Remarks. 1. Bolotov [3], considering the same problem, showed that the body will continue to roll if

$$\left|\frac{L\eta + X_1(\xi^2+\sigma)+Y_1\xi\eta}{-L\xi + X_1\xi\eta + Y_1(\sigma+\eta^2)}\right| < f \tag{1.6}$$

where $X_1 = X + m\omega^2\xi$, $Y_1 = Y + m\omega^2(\eta - (k - k_1)^{-1})$ and (X, Y) and L are the principal vector and principal moment, respectively, of the active forces when reduced to the point $G(\xi, \eta)$. Bolotov's proof of (1.6) is based on different geometrical arguments.

However, it is not difficult to rewrite condition (1.6) as a system of inequalities of type (1.5). To do this it suffices to observe that, if the constraint is strained, then $N > 0$, which is equivalent to

$$-L\xi + X_1\xi\eta + Y_1(\sigma+\eta^2) < 0 \tag{1.7}$$

Thus, Bolotov's condition is equivalent to the three inequalities (1.5) and (1.7). The theorem set out above says more: the condition for body and support to remain in contact follows logically from inequalities (1.5).

2. Most of the few theoretical publications on dry friction [3–5] are devoted to the fact that the equations of motion do not have unique solutions (the Painlevé paradox in systems with restraining couplings, the self-stopping phenomenon). There are several simple examples of systems (a homogeneous disk and a homogeneous sphere on a plane with friction) in which the alternation of rolling and sliding may be studied directly by means of easily integrable equations of motion [6].

2. ROLLING OF AN INHOMOGENEOUS WHEEL ON A HORIZONTAL RAIL

Let the support curve be a horizontal straight line ($k_1 = 0$) and let the body be a heavy inhomogeneous circular disk ($k = 1/r$). Let φ denote the angle between the downward vertical and the segment OG (where O is the geometric centre of the disk).

In dimensionless form, conditions (1.5) may be written as

$$a_+\Omega + b_+ > 0, \quad a_-\Omega + b_- < 0; \quad \Omega = \omega^2 r g^{-1} \tag{2.1}$$

where

$$a_\pm = (\xi^2+\eta^2-\eta+\sigma)(\xi \mp \eta f)+\sigma f, \quad b_\pm = -\xi\eta \pm (\eta^2+\sigma)f$$

$\xi = \rho\sin\varphi$, $\eta = 1 - \rho\cos\varphi$, g is the acceleration due to gravity, ρ is the ratio of the length of OG to r, and σ is the central moment of inertia of the body divided by mr^2.

In the translating system of coordinates Pxy (where P is the point of contact and the axis Py points vertically upward), the trajectory of the centre of mass G of the disk is a circle C

$$\xi^2 + (\eta-1)^2 = \rho^2 \tag{2.2}$$

Depending on the position of G on this circle, the numerical values of the parameters f, σ and ρ, and the value of the instantaneous angular velocity of rotation, the disk will subsequently continue to roll, begin to slide or break away from the support. The exact result depends on conditions (2.1).

Values of the parameters f, ρ and σ are specified so as to satisfy the inequality $\sigma > \xi\eta f - \xi^2$, $(\xi, \eta) \in C$.

As shown in [3], if this inequality fails to hold in a system under conditions of sliding, shock reactions may occur and, in the case of a restraining constraint, the solution may not be unique (Painlevé's paradox).

Investigating the curve

$$(x^2+y^2-y+\sigma)(x-yf)+\sigma f=0 \tag{2.3}$$

by the Newton–Puisex method [7], we obtain an asymptotic expansion

$$y=\frac{x}{f}+\frac{\sigma f^2}{1+f^2}\frac{1}{x^2}+\ldots$$

from which it follows that, if $|x|$ is large enough, the curve (2.3) will lie above its asymptote $x-yf=0$. We also observe that, for any values of the parameters, the curve will always cut the OY axis at the point $y=1$ and the OX axis at a single point $f<x_0<0$.

The form of the curve (2.3) for $f=0.4$ and a few values of σ is shown in Fig. 2. For small σ an additional closed branch may appear (this case is not considered). The curve

$$-xy+(y^2+\sigma)f=0 \tag{2.4}$$

is a hyperbola whose branches are centrally symmetric about the origin P and have asymptotes $y=0$ and $x-yf=0$; at no parameter values do the curves (2.3) and (2.4) intersect, since the left-hand side of (2.3) is equal to $(x^2+y^2+\sigma-y)\sigma fy^{-1}+\sigma f=(x^2+y^2+\sigma)\sigma fy^{-1}\neq 0$.

Note that conditions (2.1) are invariant to the replacement of ξ by $-\xi$. It will therefore suffice to analyse them for $\xi\geqslant 0$.

In the right half-plane $x>0$, the line $a_+(x)=0$ is represented by the curve (2.3) and the line $b_+(x)$ by the right branch of the hyperbola (2.4), whose left-most point has the abscissa $2f\sigma^{1/2}$. The line $a_-(x)=0$ is obtained by symmetric reflection in the Py axis of the part of the curve (2.3) that lies in the left half-plane. Finally, the curve $b_-(x)$ is obtained by the same reflection of the left branch of the hyperbola (2.4).

Depending on whether the lines $b_+(x)=0$ and $a_-(x)=0$ intersect or not, the above four curves divide the half-plane $x>0$ into five or six domains, respectively (Fig. 3 shows the case ($f=0.3$, $\sigma=0.1$, $\rho=0$)). Put

$$\Omega_1=-b_+/a_+,\quad \Omega_2=-b_-/a_- \tag{2.5}$$

Suppose that the instantaneous velocity of the point P of the disk is zero. If the centre of mass is in domain 1, then $a_+>0$, $b_+>0$, $a_-<0$, $b_-<0$. Consequently, conditions (2.1) are satisfied and there can be no sliding, whatever the angular velocity of rotation of the disk.

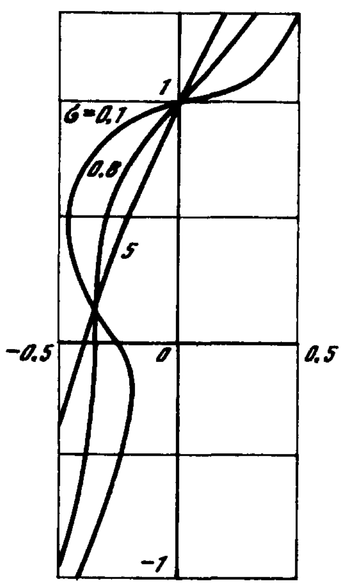

Fig. 2.

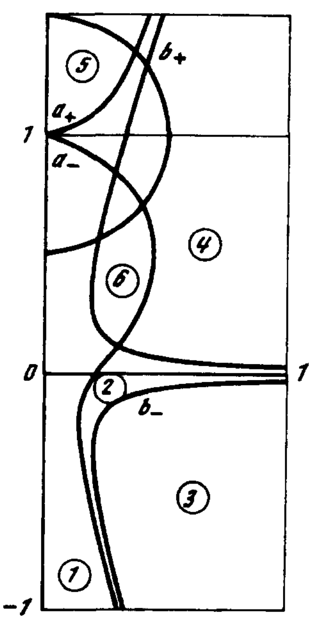

Fig. 3.

In domain 2 we have $a_+ > 0, b_+ > 0, a_- > 0, b_- < 0$. Pure rolling is maintained if the angular velocity of the disk is not too large $\Omega < \Omega_2$.

In domain 3, $a_+ > 0, b_+ > 0, a_- > 0, b_- > 0$. The first inequality in (2.1) is satisfied, the second is not. When the centre of mass is in domain 3 the disk necessarily begins to slide.

When the centre of mass is in domain 4, pure rolling is maintained provided that $\Omega_1 < \Omega < \Omega_2$ (it may be verified directly that the inequality $\Omega_1 < \Omega_2$ is always true). In domain 6 the condition for rolling to continue is expressed by a single inequality, $\Omega > \Omega_1$.

Finally, in domain 5, i.e. when the centre of mass of the disk is in a sufficiently high position, we have $a_+ < 0, b_+ > 0, a_- > 0, b_- < 0$. Conditions (2.1) take the form $\Omega < \Omega_1$ and $\Omega < \Omega_2$. In that situation we have $\Omega_1 > \Omega_2$, i.e.

$$\frac{\xi\eta-(\eta^2+\sigma)f}{(\xi^2+\eta^2-\eta+\sigma)(\xi-\eta f)+\sigma f} > \frac{\xi\eta+(\eta^2+\sigma)f}{(\xi^2+\eta^2-\eta+\sigma)(\xi+\eta f)-\sigma f}$$

(this is indeed true, since it can be reduced by equivalent transformations to the obviously true relation $(\sigma+\xi^2+\eta^2)2\xi\eta\sigma f > 0$). Consequently, rolling will be maintained if $\Omega < \Omega_2$. Conversely, if the angular velocity is large, $\Omega > \Omega_1$, both conditions (2.1) fail to hold and the constraint is weakened. In the interval $\Omega_2 < \Omega < \Omega_1$ the disk begins to slide without breaking away from the support.

When $\xi < 0$ the pattern of alternating rolling and sliding is completely symmetric.

The conditions for rolling to continue could have been expressed not in terms of Ω, which is the square of the dimensionless angular velocity of the disk, but in terms of the constant h of the energy integral $(\xi^2+\eta^2+\sigma)\Omega = 2(h-\eta)$. An attempt to do this was made, on the assumption that $\rho > 1$, in [8], but because of analytical difficulties, only the solution of the special case, when a (disk-shaped) pendulum swings at an ever-increasing amplitude and begins to slide after reaching a position of maximum deviation from equilibrium, was obtained. In the notation used here, this position corresponds to the point at which the circle (2.2) intersects the boundary of domain 3.

3. ROTATION ABOUT THE VERTICAL OF A BODY WITH CYLINDRICAL SUPPORTS, UNDER THE ACTION OF A PERMANENT TURNING MOMENT

Suppose that the support is a vertically positioned stationary circular cylinder of radius r_1 on which a rigid body is "threaded" through its cylindrical hole of radius $r > r_1$. We shall refer to this body as a "washer". A body with pins of radius r, inserted in two stationary and vertically positioned coaxial cylindrical holes of radius $r_1 > r$, will be called a "roller". The centre of the support will be denoted by O_1 and the centre of the hole (pin) in the body by O. A couple reduced to a vertical moment of constant magnitude M is applied to the body.

In dimensionless form, conditions (1.5) will be written as (2.1), where

$$a_\pm = (\xi^2+\eta^2-\eta l+\sigma)(\xi\mp\eta f)\pm\sigma l f, \quad b_\pm = M(\eta\pm\xi f)$$

The expressions for a_+ and a_- contain one more non-dimensional parameter $l = jr_1(r_1-r)^{-1}$ ($j = 1$ for the "roller", and $j = -1$ for the "washer"), which characterizes the clearance in the coupling. For the "roller", $1 < l < +\infty$ while for the "washer", $0 < l < +\infty$. In both cases, when the clearance tends to zero the parameter l tends to $+\infty$.

The position of the body when the constraint is imposed and the body may slide is defined by two angles φ and η. If P is the point of contact between the body and the support, then, as pointed out above, the positive semi-axis Py of the moving system of coordinates Pxy will point into the moving body. The angle η is measured in the counterclockwise sense, from a fixed ray O_1H, arbitrarily chosen in the plane of the motion, to the negative semi-axis Py and the angle φ from the negative semi-axis Py to the direction OG, which is fixed in the body.

In the case of pure rolling we deduce from the condition $r_1 d\theta = -r d\varphi$ that $\theta = r(c-\varphi)/r_1$, where the constant of integration c (an additional parameter) is the angular coordinate φ of the centre of mass at the instant the negative semi-axis Py coincides with the fixed ray O_1H.

The actual coordinates of the centre of mass G of the body in the moving system Pxy are determined by the formulae $\xi = \rho\sin\varphi$, $\eta = f-\rho\cos\varphi$. Thus, inequalities (2.1) do not contain θ. This fact simplifies the analysis.

The equation of the trajectory of G in coordinates Pxy is

$$\xi^2 + (\eta - j)^2 = \rho^2 \tag{3.1}$$

We will investigate how conditions (2.1) depend on the position of the point G on this circle and the values of the parameters f, σ, ρ and the instantaneous angular velocity of the body in the case of most practical interest—when the clearance is small, i.e. $l \gg 1$.

The curve

$$(x^2 + y^2 - yl + \sigma)(x - yf) + \sigma lf = 0 \tag{3.2}$$

has an asymptotic representation

$$y = \frac{x}{f} + \frac{\sigma(lf)^2}{1+(lf)^2}\,\frac{1}{x^2} + \ldots,$$

and it contains the point (0, 1) for any parameter values.

It can be shown, using the fact that the parameters $0 < f < 1$, σ, l, are positive, that the curve in question (more precisely, its branches in the finite part of the plane) does not have singular points. The shape of the curve (3.2) for $f = 0.4$, $\sigma = 1$ and $l = 4, 7, 9$, and 11 is shown in Fig. 4. A curious phenomenon is the closed branch that appears in the first quadrant as the clearance is reduced.

This fact is easily proved analytically. Rewriting Eq. (3.2) in the form

$$y^3 - (l + xf^{-1})y^2 + (\sigma + x^2 + xlf^{-1})y - (x^3 + \sigma x)f^{-1} - \sigma l = 0$$

and carrying out the standard substitution $y = z + (l + xf^{-1})/3$, we obtain the equation $z^3 + pz + q = 0$. It is well known that this equation has exactly one real root if the discriminant is such that $-4p^3 - 27q^2 < 0$, and three roots if the inverse (strict) inequality holds. We have

$$(4p^3 + 27q^2)f^4 = 4(f^2+1)^2x^6 - 8fl(f^2+1)x^5 +$$
$$+[(12\sigma - l^2)f^4 + 2(8\sigma + l^2)f^2 + 4\sigma \;\; -l^2]x^4 + 2lf[(10\sigma + l^2)f^2 - 2\sigma + l^2]x^3 +$$
$$+f^2[4\sigma(3\sigma - 5l^2)f^2 + 8\sigma^2 - 4\sigma l^2 \;\; -l^4]x^2 + 4f^3\sigma l(7\sigma - l^2)x + 4f^4\sigma(\sigma + l^2)^2$$

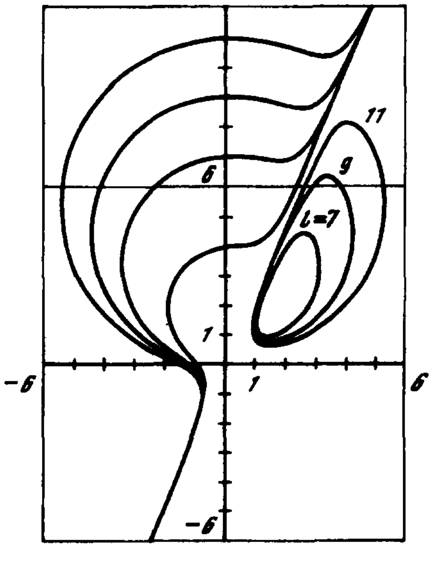

Fig. 4.

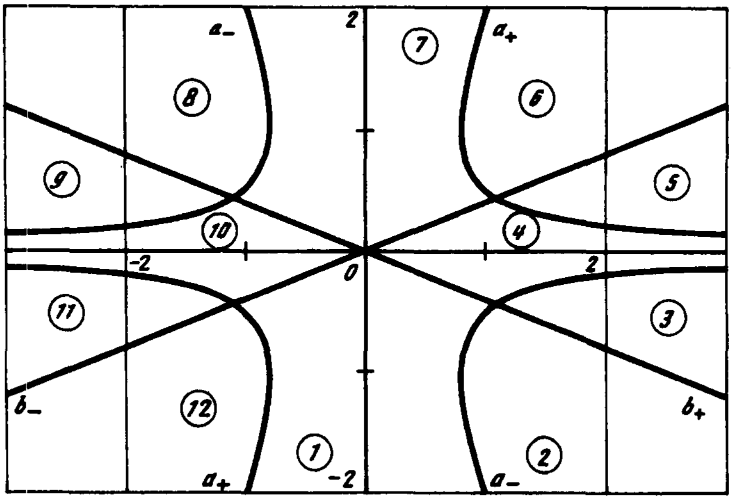

Fig. 5.

Consequently, for fixed values of the parameters σ and f, when l is allowed to increase without limit, an additional branch appears in the x-finite part of the plane in which $x^2 > 4\sigma f^2$. Moreover, for sufficiently large but finite values of l, this branch will be closed. It will lie in the first quadrant since, if we rewrite (3.2) as a polynomial in x and calculate the discriminant of the corresponding reduced equation, it is equal to $4y^3l^3 + O(l^2)$. When $l \gg 1$, therefore, Eq. (3.2) has three roots $x(y)$ only if $y > 0$. Moreover, two of them are greater than $yf < lf$, since for fixed values of $y < 1$ and the parameters, the left-hand side of (3.2) changes sign twice when x increases from yf to infinity. At $l \gg 1$ Eq. (3.2) gives two branches $-y^2 + xyf^{-1} - \sigma + O(1/l) = 0$ of the curve. The branch in the first quadrant is created from the closed loop described above. The branch in the third quadrant represents the part of the curve $a_+(x, y) = 0$ lying below the Px axis. Its other part goes to infinity.

The curve $-y^2 + xyf^{-1} - \sigma = 0$ has two asymptotes $y = 0$ and $y = xf^{-1}$. Its points nearest the Py axis have abscissae $x = \pm 2f\sigma^{1/2}$.

Figure 5, plotted for the case $l = 1000, f = 0.4$ and $\sigma = 1$, shows the curves $a_+(x, y) = 0$, $a_-(x, y) = 0$ and straight lines $b_+(x, y) = 0$, $b_-(x, y) = 0$, which divide the Pxy plane into 12 domains. By analogy with what was done in Section 2, it is not difficult to formulate the conditions for pure rolling to be maintained. The final result is as follows.

Using the notation (2.5) we find that $\Omega_1 \to 0, \Omega_2 \to 0$ as $l \to +\infty$. Therefore, for very small clearances, conditions of the type $\Omega > \Omega_1$ or $\Omega_1 < \Omega < \Omega_2$ are unrealizable in practice. For that reason, if the centre of mass of the body lies inside the domains bounded by the branches of the hyperbolas, and $\Omega \neq 0$, the body will slide. Let $M > 0$. Then rolling will be maintained in domain 1 when $\Omega > \Omega_1$ and in domain 2 when $\Omega > \Omega_2$. If $M < 0$ the inequalities are interchanged.

Thus, in the context of the mechanical model considered here, for small clearances a fairly well balanced body (e.g. when $\rho < 2f\sigma^{1/2}$), both "roller" and "washer" may be accelerated (or decelerated) to the required value of angular velocity without expending mechanical energy to overcome sliding friction at the supports. Moreover, when there is no active external torque the body will continue to roll without sliding. The principal technical difficulty is to implement the no-sliding condition at an arbitrary instant of time when $\omega \neq 0$.

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